Accurate Estimation of the Hurst Parameter of Long-Range Dependent Traffic Using Modified Allan and Hadamard Variances

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***Abstract*—Internet traffic exhibits self-similarity and long-range dependence (LRD) on various time scales. In this paper, we propose to use the Modified Allan Variance (MAVAR) and a Modified Hadamard Variance (MHVAR) to estimate the Hurst parameter H of the LRD traffic series or, more generally, the exponent α of data with $1/f^{\alpha}(\alpha \geq 0)$ power-law spectrum. MHVAR generalizes the principle of MAVAR, a time-domain quantity widely used for frequency stability characterization, to higher-order differences of input data. In our knowledge, this MHVAR has been mentioned in literature only few times and with little detail so far.**

The behaviour of MAVAR and MHVAR with power-law random processes and some common deter-ministic signals (viz. drifts, sine waves, steps) is studied by analysis and simulation. The MAVAR and MHVAR accuracy in estimating H is evaluated and compared to that of wavelet Logscale Diagram (LD). Extensive simulations show that MAVAR and MHVAR achieve significantly better confidence and no bias in H estimation. Moreover, MAVAR and MHVAR feature a number of other advantages, which make them valuable to complement other established techniques such as LD. Finally, MHVAR and LD are also applied to a real IP traffic trace.

***Index Terms*—Communication system traffic, fractals, fractional noise, Internet, long-range dependence, self-similarity, time domain analysis, wavelet transforms.**

I. Introduction

INTERNET traffic exhibits self-similarity and long-range dependence (LRD) [1][2]. In a self-similar random process, a dilated portion of a realization, by the scaling Hurst parameter H, has the same statistical characterization than the whole. On the other hand, LRD is usually equated to an asymptotic power-law decrease of the power spectral density (PSD) $\sim f^{\alpha}$ (for $f \rightarrow 0$) or, equivalently, of the autocovariance function. Under some hypotheses, the integral of a LRD process is self-similar with H related to α (e.g., fractional Brownian motion, integral of fractional Gaussian noise).

In literature, several algorithms have been defined to estimate H and α, giving prominent attention to methods based on wavelets [1]–[6]. In a different context, the Modified Allan Variance (MAVAR) is a well known time-domain quantity, originally proposed for frequency stability characterization of precision oscillators [7]–[11], purposely designed to discriminate noise types with power-law spectrum. Telecommunications standards (ANSI, ETSI, ITU-T) specify some network synchronization requirements in terms of Time Variance (TVAR), closely related to MAVAR [12]. MAVAR was also proposed as traffic analysis tool [13]–[15], pointing out its spectral sensitivity and superior accuracy in estimating H and α.

In this paper, we extend the scope of research [13]. MAVAR and a Modified Hadamard Variance (MHVAR) are studied by analysis and simulation. In our knowledge, this particular Hadamard variance has been mentioned in literature only few times and not treated in detail so far. Then, methods based on MAVAR or MHVAR to estimate H and α of traffic data are proposed. Extensive simulations show that MAVAR and MHVAR achieve highest confidence with no bias in H and α estimation. Both methods have been evaluated on pseudo-random LRD data series and compared to the well-established wavelet logscale diagram (LD) [1][6]. Also, the behaviour of MAVAR and MHVAR on common deterministic signals (viz. drifts, steps and sine waves) is studied. Finally, a real IP traffic trace is analyzed, providing a sound example of application.

II. Self-Similarity and Long Range Dependence

A random process $X(t)$ (e.g., cumulative packet arrivals in time interval $[0,t]$), is said to be *self-similar* (SS), with scaling parameter of self-similarity or Hurst parameter $H > 0$, $H \in \mathcal{R}$, if

$$X\left(t\right) =^{d} a^{-H} X\left(at\right) \tag{1}$$

for any $a > 0$, where $=^{d}$ denotes equality of all distributions of any finite order [1]. The class of SS processes is usually restricted to that of *self-similar processes with stationary increments* (SSSI), which are "integral" of a stationary process. For example, consider the δ-increment process of $X(t)$, defined as $Y_{\delta}(t) = X(t) - X(t-\delta)$ (e.g., packet arrivals in the last δ time units). For a SSSI process $X(t)$, $Y_{\delta}(t)$ is stationary and $0 < H < 1$ [1].

Long-range dependence (LRD) of a process is defined by an asymptotic power-law decrease of its autocovariance and PSD [1]. Let $Y(t)$ be a second-order stationary stochastic process. $Y(t)$ exhibits LRD if, equivalently, its autocovariance and two-sided PSD follow asymptotically

$$R_{Y}\left(\delta\right) \sim c_{1} \left|\tau\right|^{\alpha-1} \ \text{for}\ \tau \rightarrow +\infty, 0 < \alpha < 1 \tag{2}$$

$$S_{Y}\left(f\right) \sim c_{2} \left|f\right|^{-\alpha} \ \text{for}\ f \rightarrow 0, 0 < \alpha < 1 \tag{3}$$

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In general, a random process with non-integer power-law PSD is also known as fractional (not necessarily Gaussian) noise. SSSI processes $X(t)$ with $1/2 < H < 1$ have LRD increments $Y(t)$, with [1]

$$\alpha = 2H - 1 \tag{4}$$

Strictly speaking, H characterizes SS processes, but it is often used to label also the LRD increments of SSSI processes. In this paper, we follow this common custom: the expression "Hurst parameter of a LRD process" (characterized by α) denotes actually the parameter $H = (\alpha + 1)/2$ of its integral SSSI parent process.

By definition, LRD consists in a power-law behaviour of certain second-order statistics versus the duration τ of the observation interval. Therefore, several techniques to estimate H and α of data series supposed LRD are based on measuring the slope of a linear fit in a log-log plot.

III. The Modified Allan Variance

A. Background

In measurements of phase and frequency noise on precision oscillators, the power-law model (3) with $\alpha = 0, 1, 2, 3, 4$ is commonly verified [16][17]. Although values $\alpha \geq 1$ yield model pathologies, such as infinite variance and non-stationarity [18], this model is common, considering also that real-world measurements have finite bandwidth and duration. To circumvent such pathologies, it is useful to evaluate the variance of the M^{th} derivative (supposed stationary) of the process, equivalently to increasing the number of vanishing moments in wavelet analysis. In particular, the Allan Variance (AVAR) [19], recommended by IEEE in 1971 [20] for frequency stability characterization, is evaluated on the 2nd difference of phase samples. The structure function theory [21] gives a unifying view of time-domain quantities evaluated on the M^{th} difference of data.

Authors of [6] did mention AVAR and noticed that its definition can be rephrased in terms of the Haar wavelet. Under this perspective, the second difference in AVAR definition corresponds to the two vanishing moments of this wavelet. However, while they recognized fine qualities of AVAR for α estimation, they did not investigate it further, since their method based on Daubechies wavelets outperforms it.

The Modified Allan Variance is an improvement of AVAR and has played a prominent role in clock stability characterization since 1981 [7]–[12]. Being based on data second difference as AVAR, it converges on all power-law noise types with $\alpha < 5$ and is insensitive to data linear drift. In addition, it allows more accurate estimation of α over the full range $0 \leq \alpha < 5$ and in particular for $0 \leq \alpha \leq 1$, where AVAR fails. These fine qualities suggest its fruitful application also to LRD and self-similar traffic analysis.

B. Definition in the Time and Frequency Domains

Given an infinite sequence $\{x_k\}$ of samples of $x(t)$ with sampling period τ_0, MAVAR is defined as

$$\text{Mod}\sigma_y^2(\tau) = \frac{1}{2n^2\tau_0^2}\left\langle\left[\frac{1}{n}\sum_{j=1}^{n}(x_{j+2n} - 2x_{j+n} + x_j)\right]^2\right\rangle \tag{5}$$

where $< \bullet >$ denotes infinite-time averaging and $\tau = n\tau_0$ is the observation interval. In brief, it differs from unmodified AVAR in the internal average over n adjacent samples: for $n = 1$ ($\tau = \tau_0$), the two variances coincide.

As other variances [16][22], MAVAR can be equivalently redefined in the frequency domain. In fact, def. (5) can be rewritten as the mean square value of the output of a linear filter receiving $y(t) = x'(t)$ and with impulse response $h_{\text{MA}}(n, t)$ properly shaped, or, by Parseval, as integral of $S_y(f) \cdot |H_{\text{MA}}(n, f)|^2$. Expressions of $h_{\text{MA}}(n, t)$ and $|H_{\text{MA}}(n, f)|^2$ are given and plotted for example in [10][11] (Figs. 5.19, 5.20), for some values of n.

The transfer function $H_{\text{MA}}(n, f)$ is pass-band, with a narrow main lobe at $f \cong 0.4/\tau$. Consequently, MAVAR allows high-resolution spectral analysis by computation over τ. Analogously to the wavelet LD, H and α of LRD data can be estimated by a linear fit of $\text{Mod}\sigma_y^2(\tau)$ in a log-log plot.

C. Going Further: Hadamard Variances and Total Estimators

The *Hadamard Variance* (HVAR) was proposed by Baugh in 1971 [23] for higher-resolution spectral analysis. Based on a linear combination of $M + 1$ consecutive samples, HVAR may attain highest spectral selectivity by adjusting parameters [16]. In particular, weighting the $M+1$ samples with binomial coefficients (BC), better spectral selectivity than AVAR is achieved [23][24]. The $(M+1)$-samples BC-weighted HVAR is a variance of the M^{th} difference of data, whereas AVAR is a 3-samples 2nd-difference variance (Sec. III.A).

The *Total Variance, Modified Total Variance* and *Total Hadamard Variance* are improvements of conventional estimators of AVAR, MAVAR [25]–[27] and HVAR [24][28]. Total estimators improve confidence for largest τ, where few terms are averaged, by periodically extending the data sequence beyond its finite length. Unfortunately, total estimators suffer bias, which depends on the type of underlying noise and affects the curve slope. In practice, offsetting this bias makes cumbersome to estimate H and α.

In spite of its highest spectral resolution, HVAR is not able to discriminate $1/f^{\alpha}$ noise in range $0 \leq \alpha \leq 1$, similarly to AVAR. Therefore, a *Modified Hadamard Variance* (MHVAR) is proposed and studied in this paper. MHVAR has been derived by modifying the definition of the BC-weighted HVAR analogously to MAVAR. In our knowledge, such a modified HVAR has been mentioned in literature only few times (a.k.a. "pulsar variance") and with little detail so far [29][30].

IV. The Modified Hadamard Variance

MHVAR generalizes the principle of MAVAR to higher-order differences of input data. Given an infinite sequence of samples $\{x_k\}$ with sampling period τ_0, the MHVAR of order M (MHVAR-M) is defined as

$$\text{Mod}\sigma_{H,M}^2(\tau) = \frac{1}{M!n^2\tau_0^2}\left\langle\left[\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{M}\binom{M}{k}(-1)^k x_{j+kn}\right]^2\right\rangle \tag{6}$$

In brief, unmodified HVAR of order M is a kind of variance of the M^{th} difference of input data (but note the division by τ^2 instead of τ^{2M}). As MAVAR with AVAR, MHVAR

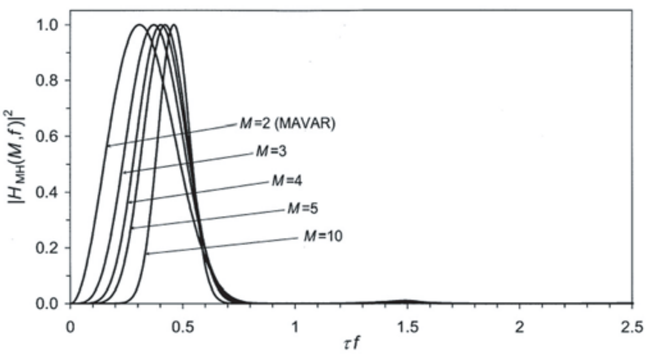

Fig. 1. Normalized square magnitude of MHVAR-M asymptotic transfer functions.

differs from HVAR in the additional internal average: for $n = 1 \quad (\tau = \tau_0)$, the two variances coincide. For $M = 2$, MHVAR coincides with MAVAR. HVAR and MHVAR of order $M = 3$ have been mostly considered in literature [16][23][24][28]–[30].

In practice, given a finite set of N samples $\{x_k\}$, MHVAR can be estimated as

$$\text{Mod}\sigma^2_{H,M}(\tau) = \frac{\sum\limits_{i=1}^{N-(M+1)n+1} \left[\sum\limits_{j=i}^{i+n-1} \sum\limits_{k=0}^{M} \binom{M}{k} (-1)^k x_{j+kn} \right]^2}{M! n^4 \tau_0^2 \left[N - (M+1)n + 1 \right]} \tag{7}$$

with $n = 1, 2, ..., \lfloor N/(M+1) \rfloor$. For $M = 2$, the estimator (7) coincides with the ITU-T standard estimator of MAVAR [11][12]. A recursive algorithm for fast computation of the MAVAR estimator exists [11], which cuts down the complexity of evaluating it for *all* $\lfloor N/3 \rfloor$ values of n to $O(N^2)$ instead of $O(N^3)$.

The point estimate (7) is a random variable itself. Exact computation of confidence intervals is not immediate and depends on the spectrum of the underlying noise [30]–[34][24]. However, in general, confidence intervals are negligible at short τ and widen for longer τ, where fewer terms are averaged. In practice, being N usually in the order of at least 10^4, $\text{Mod}\sigma^2_{H,M}(\tau)$ ripples due to poor confidence only for largest τ.

As MAVAR, MHVAR can be equivalently defined in the frequency domain. The square magnitude of the equivalent filter transfer function takes the asymptotic expression, for $n \to \infty$ and constant $n\tau_0 = \tau$:

$$\lim_{\substack{n\to\infty \\ n\tau_0=\tau}} |H_{\text{MH}}(M,n,f)|^2 = \frac{2^{2(M-1)}}{M!} \left(\frac{\sin \pi\tau f}{\pi\tau f} \right)^4 (\sin \pi\tau f)^{2(M-1)} \tag{8}$$

This limit is approached quickly for fairly low values of n (e.g., $n > 4$, cf. [11] Fig. 5.20 for MAVAR). It is plotted in Fig. 1 for some values of the parameter M, having normalized the peak magnitude to 1 and f to $1/\tau$. All these transfer functions are pass-band, with a main lobe in $0.3/\tau < f < 0.5/\tau$, as narrower as M is larger.

V. V. Behaviour of MAVAR and MHVAR

A. Power-Law Random Signals

It is convenient to extend the power-law model (3). As customary in characterization of clock phase and frequency noise [10][11][16][17][32], we deal with random processes $x(t)$ with one-sided PSD modelled as

$$S_x(f) = \begin{cases} \sum\limits_{i=1}^{P} h_{\alpha_i} f^{-\alpha_i} & 0 < f \le f_h \\ 0 & f > f_h \end{cases} \tag{9}$$

where P is the number of terms, α_i and h_{α_i} are noise parameters ($\in \mathcal{R}$) and f_h is the upper cut-off frequency. Such processes are often called *power-law* or *fractional noise*. Note that $x(t)$ is not necessarily assumed Gaussian.

Power-law noise with $0 \le \alpha_i \le 4$ has been revealed in practical measurements of various phenomena, including clock phase noise [10][11][16][17][32] and network traffic [1][2][14]. In case of simple LRD (3), then $P = 1$ and $0 < \alpha_i < 1$. Finally, the case $\alpha_i < 0$ will be not considered in this work [1].

Under this general model, first we notice that, since $|H_{\text{MH}}(M,n,f)|^2$ behaves as $\sim f^{2(M-1)}$ for $f \to 0$, MHVAR-M convergence is ensured for $\alpha_i < 1 + 2M$ (MAVAR for $\alpha_i < 5$). Also, by letting $P = 1$, $\alpha = \alpha_i$ and in the whole range $0 \le \alpha < 1 + 2M$, MHVAR-M is found to obey the power law (ideally asymptotically for $n \to \infty$, $n\tau_0 = \tau$, but in practice for $n > 4$)

$$\text{Mod}\sigma^2_{H,M}(\tau) \sim A_\mu \tau^\mu \text{ with } \mu = -3 + \alpha. \tag{10}$$

Full expressions for MAVAR ($M = 2$) are given in [10][11]. Moreover, see Rutman [16] for a detailed overview in case of unmodified variances. If $P > 1$, it is immediate to generalize (10) to $\sum\limits_i A_{\mu_i} \tau^{\mu_i}$. Therefore, if $x(t)$ obeys (9), a log-log plot of $\text{Mod}\sigma^2_{H,M}(\tau)$ looks ideally as a broken line made of P segments, whose slopes μ_i yield the estimates $\alpha_i = 3 + \mu_i$ of the fractional noise terms that are dominant in different ranges of τ.

The LD [1][6] behaves similarly. Actually, log-log plots of MHVAR(τ) can be seen as particular LDs, since modified Allan and Hadamard variances can be redefined in terms of appropriate wavelets. In [1], LDs displaying two zones with different average slopes are commented as revealing a *biscaling* phenomenon.

B. Deterministic Signals

It is of utmost interest to understand the behaviour of MAVAR and MHVAR also when $x(t)$ includes deterministic components, which are major examples of nonstationarity in Internet traffic.

1) *Offset and polynomial drift.* Let $x(t) = \sum\limits_{j=0}^{M} c_j t^j$. By substitution in (6), we get that MHVAR-M (based on the M-th difference of input data) cancels data polynomial drift of order $< M$, but reveals a $\sim t^M$ drift, then assuming trend $\sim t^{2M-2}$. Using wavelet formalism, this is explained by the fact that the MHVAR-M wavelet has M vanishing moments (cf. [6] Sec. III.B).
2) *Periodic Signals.* Let $y(t) = x'(t) = A\sin 2\pi f_{\text{m}} t$, with $S_y(f) = (A^2/2) \cdot \delta(f - f_{\text{m}})$. Then, by substitution in the frequency-domain definition, we get (for $n \to \infty$,

$n\tau_0 = \tau$):

$$\mathrm{Mod}\sigma^2_{\mathrm{H},M}(\tau) = A^2 \frac{2^{2(M-1)-1}}{M!} \cdot \left(\frac{\sin \pi f_{\mathrm{m}}\tau}{\pi f_{\mathrm{m}}\tau}\right)^4 (\sin \pi f_{\mathrm{m}}\tau)^{2(M-1)} \quad (11)$$

Hence, MAVAR and MHVAR ripple with period $2/f_{\mathrm{m}}$. (cf. [6] Sec. III.B.3)

3) *Steps*. Abrupt changes of the average bit rate in Internet traffic may be due for instance to rerouting or link capacity adjustment. Let $x(t) = Au(t)(u(t) = 0$ for $t < 0$, $u(t) = 1$ for $t \geq 0$). Since $y(t) = x'(t) = A\delta(t)$, we get

$$\mathrm{Mod}\sigma^2_{\mathrm{H},M}(\tau) = \left\langle [A \cdot h_{\mathrm{MH}}(M,n,t)]^2 \right\rangle = 0. \quad (12)$$

Thus, steps in $x(t)$ ideally do not affect MHVAR. In practice, MHVAR is estimated on a finite interval $T = (N-1)\tau_0$ and thus depends on τ and T.

VI. Using MAVAR and MHVAR for Estimating the Hurst Parameter

Let $x(t)$ be a LRD process with PSD (3) and $1/2 \leq H < 1$. Then, from (4) and (10), MAVAR and MHVAR-M follow $\sim \tau^\mu$ (ideally for $n \to \infty$) with $\mu = 2H - 4$. The following procedure is suggested to estimate H:

1) compute MAVAR/MHVAR by (7), based on $\{x_k\}$, for integer values $1 \leq n < N/(M+1)$ (we use a geometric progression of ratio 1.1, i.e. 24 values/decade, for finest rendering of trend);
2) by least-square linear regression, estimate the average slope μ of MAVAR/MHVAR in a log-log plot for $n > 4$ and excluding also highest values of n (e.g., the last decade), where confidence is lowest;
3) if $-3 \leq \mu < -2$ (i.e., $0 \leq \alpha < 1$), get the estimate of the Hurst parameter as

$$H = \mu/2 + 2 \quad (13)$$

Under the more general hypothesis of power-law PSD (9), then up to P slopes μ_i can be identified ($-3 \leq \mu_i < 2M-2$) to yield the estimates $\alpha_i = 3 + \mu_i \quad (0 \leq \alpha < 1 + 2M)$ of the P components of $f^{-\alpha_i}$ noise.

Some care should be exercised against non-stationary terms in data analyzed (e.g., steps, slow trends), which cause slope changes that may be erroneously ascribed to random power-law noise. Moreover, the order M can be conveniently adjusted to cancel polynomial drifts. Similarly, in [6] Sec. III.B.4, it is suggested to increase the number of vanishing moments until the H estimate converges to a stable value.

A key issue is to determine the confidence of these H and α_i estimates and whether they are unbiased. In [6] Sec. III.C, this problem is studied for the H estimator based on wavelet decomposition, under a number of simplifying assumptions. This analysis can be adapted to the estimator based on MAVAR or MHVAR, by defining them in terms of appropriate wavelets. However, deriving exact expressions for these confidence intervals is not immediate. Being analysis cumbersome, we chose to evaluate empirically the accuracy of the method proposed, by simulation on pseudorandom data as done for example in [4].

VII. Simulation Results

The accuracy of the MAVAR/MHVAR method was evaluated by extensive simulations and compared to that of the wavelet LD technique [1][6]. All LD results were computed running original scripts [35], using Daubechies wavelet with $N_v = 3$ vanishing moments (LD-3).

A. Accuracy Evaluation

The MHVAR-3, MAVAR and LD-3 methods were applied to LRD pseudo-random series $\{x_k\}$ of length N, generated with one-sided PSD $S_x(f) = hf^{-\alpha}(0 \leq \alpha < 1)$ for assigned values of $H = (1+\alpha)/2$. The generation algorithm is by Paxson [36]: a vector of random complex samples, having amplitude equal to the square root of an exponentially-distributed variable with mean $S_x(f_k)$ and phase uniformly distributed in $[0, 2\pi]$, is inversely Fourier-transformed to yield the time-domain sequence $\{x_k\}$. Also other synthesis algorithms were essayed, but with no substantially different results in comparing accuracy of H estimates.

First, 100 independent pseudo-random sequences $\{x_k\}$ of length $N = 131072$, with mean $m_x = 0$ and variance $\sigma_x^2 = 1$, were generated for each of the 11 values $\{H_i\} = \{0.50, 0.55, ..., 1.00\}$. On the resulting 1100 time series, we applied the MHVAR-3, MAVAR and LD-3 methods, getting three sets of estimates $\left\{\widehat{H}_{i,j}\right\}$, for $i = 0, 1, \ldots, 10$ and $j = 1, 2, \ldots, 100$. We then evaluated the accuracy of these estimates, calculating the absolute estimation errors $\Delta_{i,j} = \widehat{H}_{i,j} - H_i$. The same test was repeated on other 1100 sequences of length $N = 1024$, to compare the methods on short sequences too.

Fig. 2 compares the estimation errors $\{\Delta_{i,j}\}$ attained on series of $N = 131072$ samples. For each value H_i, the mean $m_{\Delta i}$ and standard deviation $\pm\sigma_{\Delta i}$, out of 100 estimation errors, are plotted. Both MAVAR and MHVAR-3 achieve better confidence than LD-3, which also appears significantly biased. Similarly, Fig. 3 compares the estimation errors $\{\Delta_{i,j}\}$ on series of $N = 1024$ samples. Also here, both MAVAR and MHVAR-3 achieve better confidence than LD-3, which seems less efficient on shorter data series. Visual comparison of plots, not shown for lack of space, justifies such better confidence of estimates. Especially on short series, MAVAR/MHVAR log-log plots are smoother and closer to the ideal linear trend than LD.

B. Impact of Steps in LRD Input Data

We evaluated MAVAR, MHVAR-M and LD-3 on LRD data including various steps (cf. Sec. V.B). Sequences of length $N = 1024, 131072$ were generated as $\{x_k\} = \{Au_{k-Q} + n_k\}(k = 1, \ldots, N)$, where $\{u_{k-Q}\}$ is the sampled unit step function $u(t)$ delayed Q time units $(1 < Q < N)$ and $\{n_k\}$ is a pseudo-random LRD series, with mean $m_n = 0$ and variance $\sigma_n^2 = 1$, generated as before with PSD $S_n(f) = hf^{-\alpha}$ for $\alpha = 0.60 \quad (H = 0.80)$. By varying extensively parameters Q and A, we found that:

- the step impact on MHVAR-M is maximum for $Q = N/2$ and negligible for $Q \to 1$ and $Q \to N$;

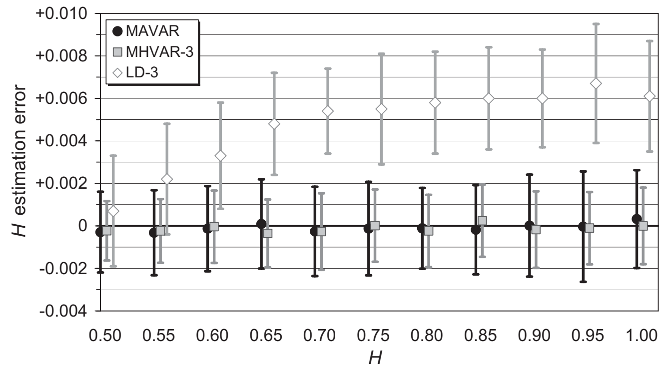

Fig. 2. Absolute estimation error of H attained by MAVAR, MHVAR-3 and LD-3 methods ($N = 131072$, mean and standard deviation out of 100 estimation errors).

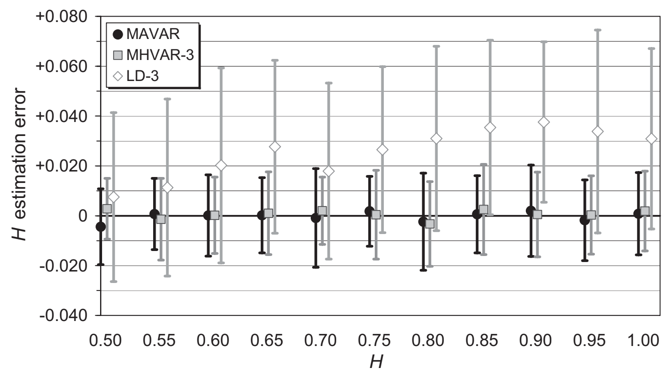

Fig. 3. Absolute estimation error of H attained by MAVAR, MHVAR-3 and LD-3 methods ($N = 1024$, mean and standard deviation out of 100 estimation errors).

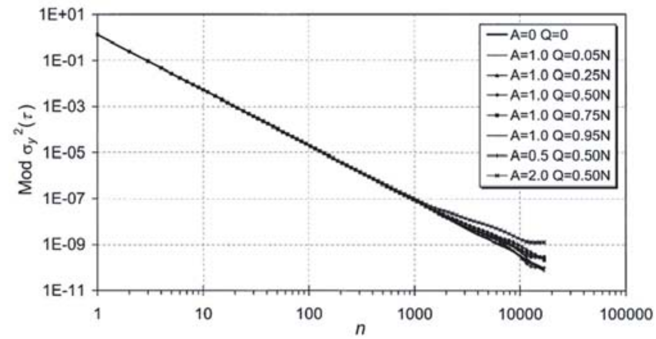

Fig. 4. MAVAR computed on a pseudo-random LRD sequence $\{n_k\}$ ($N = 131072$, $m_n = 0$, $\sigma_n = 1$, $H = 0.80$) with added step $\{A_{u_{k-Q}}\}$.

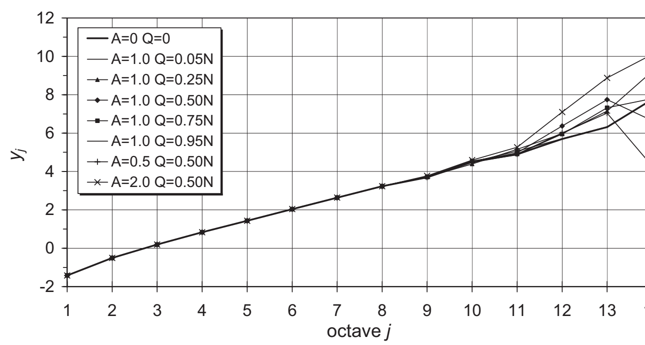

Fig. 5. LD-3 computed on a pseudo-random LRD sequence $\{n_k\}$ ($N = 131072$, $m_n = 0$, $\sigma_n = 1$, $H = 0.80$) with added step $\{A_{u_{k-Q}}\}$.

- for M even, input steps affect MHVAR-M curves only at the right end;
- for M odd, input steps shift MHVAR-M curves vertically, with limited impact on slope ([37] Fig. 4);
- MHVAR-M is affected significantly if steps have size at least on the order of σ_n; such big steps are evident by visual inspection and can be removed before H estimation;
- anyway, input steps affect MHVAR-M (M even) less than LD-3.

Among many simulation results produced, Figs. 4 and 5 show MAVAR and LD-3 curves for $N = 131072$, varying step size and delay as $0 \leq A \leq 2$ and $0 < Q < N$. Moreover, MHVAR-M curves for $M = 3$, 4 were shown in Fig. 4 of [37]. To compare MAVAR and LD graphs fairly, since LD omits the last two octaves $(j \leq 14)$ [35], MAVAR curves in Fig. 4 have been plotted for $n \leq 2^{14}$.

C. Impact of the Difference Order M

We evaluated how the order M affects the H estimation accuracy by MHVAR, on 100 independent pseudo-random sequences $\{x_k\}$ for each of the 11 values H_i as in Sec. VII.A. Fig. 6 plots, for $2 \leq M \leq 10$, the mean of the 11 mean values $m_{\Delta i}$ (dots) and of the 11 standard deviations $\sigma_{\Delta i}$ (bars) for $N = 1024$ (right) and $N = 131072$ (left). LD-3 results are also plotted for comparison, although half out of scale.

First, MAVAR and MHVAR-M estimates are not biased. Second, MHVAR-3 results have better confidence than MAVAR for $N = 131072$: the mean of $\sigma_{\Delta i}$ of MHVAR-3 estimates is -22% than that of MAVAR, which in turn is -14% than that of LD-3 (cf. Fig. 2). This confidence gain is significant, since it is computed over 1100 independent estimates. For $N = 1024$, the mean of $\sigma_{\Delta i}$ of MHVAR-3 estimates is just -3% than that of MAVAR, which is -54% than that of LD-3.

Conversely, increasing the order $M > 4$ does not improve confidence further for $N = 131072$, whereas it even worsens it for $N = 1024$. In general, the confidence is not improved by increasing the MHVAR order indefinitely, although $H_{\mathrm{MH}}(M, f)$ becomes more selective (cf. Fig. 1). In fact, the sharper is the filter transfer function, the longer is its time-domain response, and thus the longer the data sequence should be for achieving the same confidence. Analogous considerations hold for the number of vanishing moments N_V in wavelet LD. As noted in [6], the larger N_V is, the smaller are bias and confidence intervals of H estimates. Nevertheless, this improvement is counterbalanced by the increase of the number of wavelet coefficients polluted by border effects due to data finite length, resulting in a smaller number of usable wavelet coefficients.

VIII. Example of Application to a Real IP Traffic Trace

We applied the MHVAR-3 and LD-3 methods on a real IP traffic series [bytes/s] measured on a trans-oceanic link

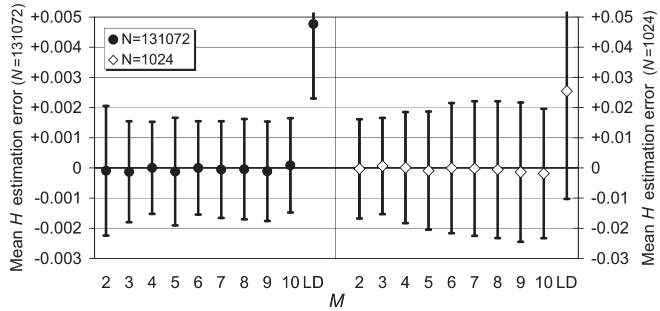

Fig. 6. Average mean $\mathrm{E}[m_{\Delta i}]$ and standard deviation $E[\sigma_{\Delta i}](i = 0, \ldots, 10)$ of the H estimation errors attained by the MHVAR-M method ($2 \leq M \leq 10$), compared to LD 3 (average results on 100 pseudo-random sequences $\{x_k\}$ for each of 11 values H_i).

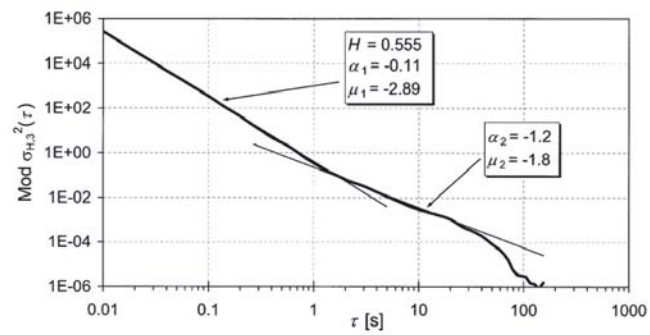

Fig. 8. Modified Hadamard Variance ($M = 3$, 24 points/decade) of a real IP traffic sequence (bytes/time, MAWI Project [38], $N = 61600$, $\tau_0 = 10$ ms, $T = 616$ s).

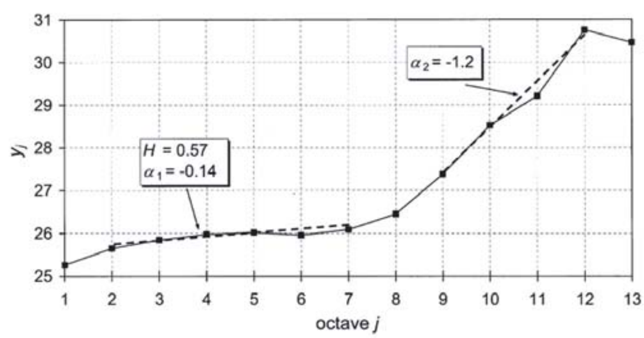

Fig. 7. Logscale diagram ($N_v = 3$) [35] of a real IP traffic sequence (bytes/time, MAWI Project [38], $N = 61600$, $\tau_0 = 10$ ms, $T = 616$ s).

(MAWI [38], $N = 61600$, $\tau_0 = 10$ ms, $T = 616$ s). No nonstationary trends, e.g. steps, are evident.

Figs. 7 and 8 plot respectively LD-3 and MHVAR-3. We notice that the LD-3 trend is more irregular (cf. Figs. 2 and 3), whereas MHVAR exhibits two regular slopes, viz. $\mu_1 = -2.89$ and $\mu_2 = -1.8$. Thus, two simple power-law (9) components are revealed by MHVAR: $\alpha_1 \cong 0.11$ ($H \cong 0.555$), dominant for 10 ms $< \tau <$ 2 s, and $\alpha_2 \cong 1.2$, dominant for 2 s $<$ $\tau <$ 20 s. Both estimates are in agreement with average slopes observed on LD-3 (note that octave 8 corresponds to $\tau \cong 2$ s).

IX. Conclusions

In this paper, the Modified Allan and Hadamard Variances have been proposed for estimating the Hurst parameter H or the exponent α of traffic series with $1/f^\alpha$ power-law PSD ($\alpha \geq 0$). While MAVAR is well known as frequency stability measure, MHVAR has been given little attention so far. Some other variances (e.g., total variances) were also studied, but resulted less useful to the purpose of H and α estimation.

The H estimation accuracy of MAVAR and MHVAR-M was evaluated on LRD pseudo-random sequences, generated with assigned values of H, and compared to the Daubechies' wavelet LD technique with 3 vanishing moments. Extensive simulations showed that MAVAR and MHVAR-M achieve significantly better confidence and are not biased in H estimation. The behaviour of MAVAR and MHVAR-M with drifts, steps and periodic components in input data was also investigated. Being based on data M^{th} difference ($M \geq 2$), they cancel $\sim t^{M-1}$ drifts. Moreover, they are quite robust against steps, being affected less than LD-3.

Finally, MHVAR-3 was applied to a real IP traffic trace. Compared to LD-3, MHVAR-3 gave a clearer spectral characterization of the traffic series analyzed. Two power-law noise components were identified ($k_1/f^{0.11} + k_2/f^{1.2}$), revealing two different scaling behaviours dominant on different observation intervals.

In conclusion, MAVAR and MHVAR-M may complement usefully other well-established techniques (e.g. LD), due to several advantages. Among them, we highlight in particular:

- high spectral resolution (Fig. 1);
- excellent accuracy in H and α estimation with negligible bias (Figs. 2, 3 and 6);
- convergence to finite values for all types of $1/f^\alpha$ processes (9) with $\alpha < 1 + 2M (\alpha \in \mathcal{R})$;
- insensitivity to polynomial drifts of order $\leq M - 1$ and robustness against steps and periodic components;
- affordable computational complexity in all practical cases, since MAVAR and LD have same complexity $O(N \log N)$ if MAVAR is computed by recursive algorithm for $n = 2^j$, i.e. on octaves as LD;
- ease of computation for *any* value of $\tau = n\tau_0 (n = 1, 2, ..., \lfloor N/(M+1) \rfloor)$, which allows rendering the trend to the finest detail (contrary to LD, which is computed on octaves instead).

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